ANALYSIS OF WEAK DISCONTINUITIES IN MAGNETOHYDROMECHANICS

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1. Assumptions. By a weak discontinuity we mean a surface on which all elements characterizing the state of a stream of gas are continuous, but their first or higher derivatives undergo discontinuities. We impose no limitation whatsoever on the order of magnitude of the discontinuities, except that they are finite.

We suppose the medium in which we study weak discontinuities to be ideal, that is, devoid of processes for dissipating energy (no Joule losses, which corresponds to infinite conductivity of the medium, and absence of internal friction and heat conduction). The density ρ, pressure p and entropy per unit mass are assumed to be related by an equation of state of the general form $p = f(\rho, s)$.

The equations of magnetohydromechanics for an ideal medium have the form (see, for example, [1]):

$$\operatorname{div} \mathbf{H} = 0, \qquad \frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}(\mathbf{v} \times \mathbf{H}), \qquad \frac{ds}{dt} = 0$$

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} p - \frac{1}{4\pi}[\mathbf{H} \times \operatorname{rot} \mathbf{H}], \qquad \frac{\partial \rho}{\partial t} + \operatorname{div} \rho \cdot \mathbf{v} = 0 \qquad (1.1)$$

where **H** is the vector intensity of the magnetic field, **v** the vector velocity, and the symbol d/dt denotes the full derivative with respect to time for fixed particles.

2. Kinematic conditions on a weak discontinuity. Let the surface $\phi(x, y, z, t) = 0$ be a surface of weak discontinuity, and let the function $u(x, y, z, t)$ be continuous at this surface, but its first derivatives suffer discontinuities across it. We then form two functions u_1 and u_2 such that the function u_1 coincides with u on one side and the function u_2 coincides with u on the other side of the survace $\phi = 0$, with the functions u_1 and u_2 determined on both sides of the surface $\phi = 0$ and continuous at it together with their first derivatives.

Forming the difference $u_2 - u_1$ and differentiating it along $\phi(x, y, z, t) = 0$, we obtain

$$\left[\frac{\partial u}{\partial x}\right] dx + \left[\frac{\partial u}{\partial y}\right] dy + \left[\frac{\partial u}{\partial z}\right] dz + \left[\frac{\partial u}{\partial t}\right] dt = 0 \qquad ([\alpha] = \alpha_2 - \alpha_1) \tag{2.1}$$

Here the differentials dx, dy, dz, dt are subject to the single condition

$$\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz + \frac{\partial \varphi}{\partial t} dt = 0 \tag{2.2}$$

From a comparison of expressions (2.1) and (2.2) it follows that

$$\left[\frac{\partial u}{\partial x}\right] : \frac{\partial \varphi}{\partial x} = \left[\frac{\partial u}{\partial y}\right] : \frac{\partial \varphi}{\partial y} = \left[\frac{\partial u}{\partial z}\right] : \frac{\partial \varphi}{\partial z} = \left[\frac{\partial u}{\partial t}\right] : \frac{\partial \varphi}{\partial t} = \mu_u (x, y, z, t) \tag{2.3}$$

Thus the kinematic condition (2.3) at a weak discontinuity consists in the fact that giving the discontinuity in one of the first derivatives of a function $u(x, y, z, t)$ across a surface $\phi(x, y, z, t)$ is necessary and sufficient for determining all the other first derivatives. Thus a weak discontinuity in any quantity $u(x, y, z, t)$ can be characterized by a single function $\mu_u(x, y, z, t)$ with

$$\left[\frac{\partial u}{\partial x}\right] = \mu_u \frac{\partial \varphi}{\partial x}, \qquad \left[\frac{\partial u}{\partial t}\right] = \mu_u \frac{\partial \varphi}{\partial t}$$

or

$$\left[\frac{\partial u}{\partial x}\right] = \lambda_u \mathbf{n}_x, \qquad \left[\frac{\partial u}{\partial t}\right] = -\lambda_u \mathbf{N} \tag{2.4}$$

where $\mathbf{n}$ is the unit vector normal to the surface $\phi(x, y, z, t) = 0$ and $\mathbf{N}$ is the velocity of propagation of the discontinuity:

$$\mathbf{N} = -\mathbf{n} \frac{\partial \varphi}{\partial t} \Big/ \sqrt{\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2}$$

On the basis of (2.4) the following expressions may easily be obtained:

$$[\mathrm{rot}\,(\mathbf{v} \times \mathbf{H})] = \mathbf{n} \times (\boldsymbol{\lambda}_{\mathbf{v}} \times \mathbf{H}) - \mathbf{n} \times (\boldsymbol{\lambda}_{\mathbf{H}} \times \mathbf{v}), \qquad [\mathrm{div}\,(\rho \cdot \mathbf{v})] = \rho\,(\boldsymbol{\lambda}_{\mathbf{v}} \cdot \mathbf{n}) - \lambda_\rho v_n \tag{2.5}$$

$$\left[\frac{d\mathbf{v}}{dt}\right] = -\theta \boldsymbol{\lambda}_{\mathbf{v}}, \qquad [\mathrm{grad}\; p] = \lambda_p \mathbf{n}, \qquad [\mathbf{H} \times \mathrm{rot}\, \mathbf{H}] = \mathbf{H} \times (\mathbf{n} \times \boldsymbol{\lambda}_{\mathbf{H}})$$

Here

$$\boldsymbol{\lambda}_{\mathbf{v}} = \lambda_{v_x} \mathbf{i} + \lambda_{v_y} \mathbf{j} + \lambda_{v_z} \mathbf{k}, \qquad \theta = N - v_n, \qquad v_n = (\mathbf{v} \cdot \mathbf{n})$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors in the Cartesian coordinate system x, y, z.

3. Dynamic conditions on a discontinuity. We obtain dynamical conditions on the discontinuity, or a relation between the quantities λ_i, by using expression (2.5) and the equations of magnetohydromechanics for the differences across the discontinuity. We will have

$$
\begin{gathered}
\boldsymbol{\lambda}_H \cdot \mathbf{n} = 0, \qquad \boldsymbol{\lambda}_H \theta + \mathbf{n} \times (\boldsymbol{\lambda}_v \times \mathbf{H}) = 0 \\
\boldsymbol{\lambda}_v \theta - \frac{1}{\rho} \lambda_p \mathbf{n} - \frac{1}{4\pi\rho} \mathbf{H} \times (\mathbf{n} \times \boldsymbol{\lambda}_H) = 0 \\
\lambda_\rho \theta - \rho(\boldsymbol{\lambda}_v \cdot \mathbf{n}) = 0 \\
\lambda_s \theta = 0, \qquad \lambda_p = a^2 \lambda_\rho + q \lambda_s \qquad \left(a^2 = \frac{\partial f}{\partial \rho}, \quad q = \frac{\partial f}{\partial s}\right)
\end{gathered}
\tag{3.1}
$$

The last of equations (3.1) is obtained by differentiating the equation $p = f(\rho, s)$; and a is the speed of sound. The first of equations (3.1) indicates that derivatives of the magnetic field intensity along the normal to the discontinuity are continuous across the surface of weak discontinuity.

Expressing λ_H from the second and fourth of equations (3.1) in the form

$$
\boldsymbol{\lambda}_H = -\frac{\boldsymbol{\lambda}_v}{\theta} H_n + \mathbf{H} \frac{\lambda_\rho}{\rho}
\tag{3.2}
$$

and substituting into the third equation, allowing also for the sixth of equations (3.1), we have

$$
\boldsymbol{\lambda}_v \left(\theta - \frac{b_n^2}{\theta}\right) - \frac{\lambda_\rho}{\rho} (a^2 \mathbf{n} + \mathbf{b} \times (\mathbf{n} \times \mathbf{b})) - \frac{q}{\rho} \lambda_s \mathbf{n} + \frac{b_n}{\theta} (\boldsymbol{\lambda}_v \cdot \mathbf{b})\, \mathbf{n} = 0
\tag{3.3}
$$

(Here $\mathbf{b} = \mathbf{H} / 2\sqrt{\pi\rho}$).

Taking the scalar product of equation (3.3) by $\mathbf{b}$, we find

$$
\boldsymbol{\lambda}_v \cdot \mathbf{b} = \frac{a^2 b_n}{\theta} \frac{\lambda_\rho}{\rho} + \frac{q b_n}{\theta} \frac{\lambda_s}{\rho}
\tag{3.4}
$$

and we substitute this back into (3.3). Thus a system of equations equivalent to the system (3.1) is

$$
\begin{gathered}
\boldsymbol{\lambda}_H \cdot \mathbf{n} = 0, \quad \boldsymbol{\lambda}_H = -\frac{\boldsymbol{\lambda}_v}{\theta} H_n + \mathbf{H} \frac{\lambda_\rho}{\rho}, \quad \lambda_\rho \theta - \rho \boldsymbol{\lambda}_v \cdot \mathbf{n} = 0 \\
\boldsymbol{\lambda}_v \left(\theta - \frac{b_n^2}{\theta}\right) = \frac{\lambda_\rho}{\rho} \left(a^2 \mathbf{n} \left(1 - \frac{b_n^2}{\theta^2}\right) + \mathbf{b} \times (\mathbf{n} \times \mathbf{b})\right) + q \mathbf{n} \frac{\lambda_s}{\rho} \left(1 - \frac{b_n^2}{\theta^2}\right) \\
\lambda_s \theta = 0
\end{gathered}
\tag{3.5}
$$

The last of equations (3.1) now serves only for the determination of λ_p.

Equations (3.1) were obtained under the assumption that the first derivatives are discontinuous. But a case may arise where higher derivatives suffer discontinuities. Thus, we assume as an example that all elements of the stream and their first derivatives are continuous, but the second derivatives are discontinuous across the surface $\phi(x, y, z, t) = 0$. Differentiating (with respect to x, for example) all the equations of magnetohydromechanics, and forming their differences across the surface of discontinuity, we obtain a system of equations for the quantities λ_i just line (3.1), where the quantities λ_v, λ_H, λ_ρ and so on are now introduced according to the following relations

$$\left[\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{v}}{\partial x}\right)\right] = \boldsymbol{\lambda}_{\mathbf{v}} n_y, \qquad \left[\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{H}}{\partial x}\right)\right] = \boldsymbol{\lambda}_{\mathbf{H}} n_y, \qquad \left[\frac{\partial}{\partial y}\left(\frac{\partial \rho}{\partial x}\right)\right] = \lambda_\rho n_y \text{ и т. д.}$$

The situation is similar for discontinuous derivatives of any order. The present reasoning displays the fundamental role of system (3.1) or system (3.5) in magnetohydromechanics.

The system of equations (3.5) is a linear homogeneous system of eighth order with respect to the quantities λ_i. Consequently, λ_i different from zero exist only in the case when the determinant of the system (3.5) vanishes. This last condition determines the value θ of the speed of propagation of the discontinuity with respect to the particles of gas.

Calculating the determinant of system (3.5) and equating it to zero, we obtain

$$\theta^2(\theta^2 - b_n^2)[\theta^4 - \theta^2(a^2 + b^2) + a^2 b_n^2] = 0 \tag{3.6}$$

According to the character of the dependence of θ on the elements of the gas stream, and adopting the terminology for a plane wave of infinitesimally small amplitude in the stream $\mathbf{v} = \text{const}_1$, $\mathbf{H} = \text{const}_2$, weak discontinuities in magnetohydromechanics can be classified in the following way:

magnetohydrodynamic discontinuity:

$$\theta^2 = b_n^2 \tag{3.7}$$

magnetoacoustic discontinuity:

$$\theta^4 - \theta^2(a^2 + b^2) + a^2 b_n^2 = 0 \tag{3.8}$$

entropy discontinuity:

$$\theta = 0 \tag{3.9}$$

4. **Magnetohydrodynamic discontinuity.** In this case the speed of propagation $\theta = \pm b_n$. Choosing as the x axis the direction of the vector $\mathbf{n}$, and as the x, y plane the plane containing the vectors $\mathbf{n}$ and $\mathbf{H}$, on the

basis of the second and fifth equations of system (3.5) we have

$$\lambda_{v_z} = \mp b_n \lambda_{H_z}, \qquad \lambda_s = 0 \tag{4.1}$$

From the remaining equations of system (3.5) it follows that when vectors **n** and **H** are not parallel:

$$\lambda_\rho = \boldsymbol{\lambda}_{\mathbf{v}} \cdot \mathbf{n} = \boldsymbol{\lambda}_{\mathbf{v}} \cdot \mathbf{H} = \boldsymbol{\lambda}_{\mathbf{H}} \cdot \mathbf{n} = \boldsymbol{\lambda}_{\mathbf{H}} \cdot \mathbf{H} = 0 \tag{4.2}$$

However, if vectors **n** and **H** are parallel, and moreover $a^2 = b^2$, then on the basis of (3.4) $\lambda_H \cdot \mathbf{n} = 0$, but $\lambda_v \cdot \mathbf{n}$ is arbitrary, and generally speaking it is different from zero and λ_p (here the z axis can be taken in any direction perpendicular to **n**).

We now consider steady flow. Owing to the fact that in this case the discontinuity surface is stationary, $\theta = -v_n$; consequently

$$(\mathbf{v} \pm \mathbf{b}) \cdot \mathbf{n} = 0 \tag{4.3}$$

The expression (4.3) can be considered as the equation for possible positions of a surface of weak discontinuity at a given point.

On the basis of (4.3) it is evident that in the steady case a magnetohydrodynamic discontinuity surface is oriented so that its normal is orthogonal either to the vector $\mathbf{v} + \mathbf{H}/2\sqrt{\pi\rho}$ or to the vector $\mathbf{v} - \mathbf{H}/2\sqrt{\pi\rho}$.

5. **Magnetoacoustic discontinuity.** For the functions λ_i, on the basis of (3.5), we obtain the following relations

$$\lambda_s = 0, \qquad \boldsymbol{\lambda}_{\mathbf{v}} = \left(\frac{a^2}{\theta}\mathbf{n} + \frac{\theta \mathbf{b} \times (\mathbf{n} \times \mathbf{b})}{\theta^2 - b_n^2}\right)\frac{\lambda_\rho}{\rho}$$
$$\lambda_p = a^2 \lambda_\rho, \qquad \boldsymbol{\lambda}_{\mathbf{H}} = 2\sqrt{\pi\rho}\left(\mathbf{b} - \frac{b_n a^2}{\theta^2}\mathbf{n} - \frac{b_n \mathbf{b} \times (\mathbf{n} \times \mathbf{b})}{\theta^2 - b_n^2}\right)\frac{\lambda_\rho}{\rho} \tag{5.1}$$

We first consider the unsteady case, regarding **n** as a given quantity. Solving equation (3.8) with respect to θ^2, we obtain

$$2\theta_{\alpha,\beta}^2 = (a^2 + b^2) \pm \sqrt{(a^2 + b^2)^2 - 4a^2 b_n^2} \tag{5.2}$$

An obvious consequence is the estimates

$$\max(a^2, b^2) \leqslant \theta_\alpha^2 \leqslant a^2 + b^2, \qquad 0 \leqslant \theta_\beta^2 \leqslant \min(a^2, b^2) \tag{5.3}$$

Here $\max(a^2, b^2)$ and $\min(a^2, b^2)$ indicate quantities which are the maximum and minimum among the values of a^2 and b^2. We choose the direction of the velocity **v** as the x axis, the plane of the vectors **v** and **H** as the x, y plane; then equation (3.8) can be rewritten as

$$v^4 l^4 - v^2 l^2 (a^2 + b^2) + a^2 b^2 (\cos\gamma l + \sin\gamma m)^2 = 0 \tag{5.4}$$

Here $l = \cos(\mathbf{n}, x)$, $m = \cos(\mathbf{n}, y)$, and γ denotes the angle between the directions of $\mathbf{v}$ and $\mathbf{H}$. Solving equation (5.4) with respect to m, we obtain

$$m = -\frac{l}{\sin\gamma}\left(\cos\gamma \pm \frac{v}{ab}\sqrt{a^2 + b^2 - v^2 l^2}\right) \tag{5.5}$$

Expression (5.5) can be considered as an equation for the determination of possible elements of a surface of weak discontinuity. The totality of these elements lie on planes tangent to a certain conical surface with vertex at the point in question. On the basis of (5.3) and (5.5) it follows that the condition

$$v^2 = a^2 + b^2 \tag{5.6}$$

is sufficient for the conical surface under consideration to exist. Estimates (5.3) show that, generally speaking, the conical surface for the case of supersonic flow consists of two concentric surfaces: one located outside the Mach cone ($v^2 l^2 = a^2$) and inside the circular cone $v^2 l^2 = a^2 + b^2$, and the other situated inside the Mach cone.

In the special case when vectors $\mathbf{v}$ and $\mathbf{H}$ are parallel, the first surface is the circular cone $v^2 l^2 = a^2 + b^2 - a^2 b^2 / v^2$, and the second degenerates to the straight line $l = 0$. The condition for the existence of a surface of weak discontinuity in this case is obviously

$$0 \leqslant l^2 \leqslant 1$$

or

$$\begin{aligned} &\text{either} \quad v^2 > a^2, \quad b^2 \leqslant v^2 \\ &\text{or} \quad v^2 < a^2, \quad v^2 \leqslant b^2 \leqslant \frac{a^2 v^2}{a^2 - v^2} \\ &\text{or} \quad v^2 = a^2, \quad b^2 \text{— arbitrary} \end{aligned} \tag{5.7}$$

6. Entropy discontinuity. For an entropy discontinuity $\theta = 0$, that is, it moves with the gas particles, and in steady flow coincides with a stream surface. From the fourth equation of the system (3.1) it follows that $\lambda_{\mathrm{v}} \cdot \mathbf{n} = 0$, and then on the basis of the second of equations (3.1)

$$\lambda_{\mathrm{v}} H_n = 0 \tag{6.1}$$

If $H_n \neq 0$, then $\lambda_{\mathrm{v}} = 0$, and from the third equation of system (3.1) it follows that $\lambda_{\mathrm{H}} = \lambda_{\mathrm{p}} = 0$, but then

$$\lambda_\rho = -(q/a^2)\lambda_s \tag{6.2}$$

If $H_n = 0$, then

$$\lambda_p = -\frac{1}{4\pi}\boldsymbol{\lambda}_{\mathrm{H}}\cdot\mathrm{H}, \qquad \boldsymbol{\lambda}_{\mathrm{H}}\cdot\mathbf{n} = 0, \qquad \boldsymbol{\lambda}_{\mathbf{v}}\cdot\mathbf{n} = 0.$$
$$\lambda_\rho = \frac{\lambda_p}{a^2} - \frac{q\lambda_s}{a^2} \tag{6.3}$$

Thus in the last case ($H_n = 0$), a derivative formed from the velocity and magnetic field intensity may suffer a discontinuity in a surface tangent to the surface of discontinuity.

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